Homework based on Chapter 5
Computational Probability and Statistics
CIS 2033, Section 002

1 Part 1 (Due: 9:00 AM, Friday, Feb. 13, 2015)

Question 1 Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{7}{8} & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \frac{1}{16} & \text{for } 3 \leq x \leq 5 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Draw the graph of f.

b. Determine the distribution function of F of X, and draw its graph.

Question 2 The probability density function f of a continuous random variable X is given by:

$$f(x) = \begin{cases} C(4x - 2x^2) & \text{for } 0 \leq x \leq 2 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Compute c.

b. Compute the distribution function F of X.

c. Compute $F(1)$ and $P(X > 1)$

2 Part 2 (Due: 11:59 PM, Tuesday, Feb. 17, 2015)

Question 3 Suppose we choose arbitrarily a point from the square with corners at $(2,1),(3,1),(2,2),(3,2)$. The random variable A is the area of the triangle with its corners at $(2,1),(3,1)$ and the chosen point.

a. What is the largest area A that can occur, and what is the set of points for which $A \leq \frac{1}{4}$?

b. Determine the distribution function F of A.

c. Determine the probability density function f of A.

Question 4 Compute the median of an $Exp(3)$ distribution.

Question 5 Compute the median of a $Par(1)$ distribution.

Appendix

5.1 Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{3}{4} & \text{for } 0 \leq x \leq 1 \\ \frac{1}{4} & \text{for } 2 \leq x \leq 3 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Draw the graph of f. The Probability density function is in Figure 1.

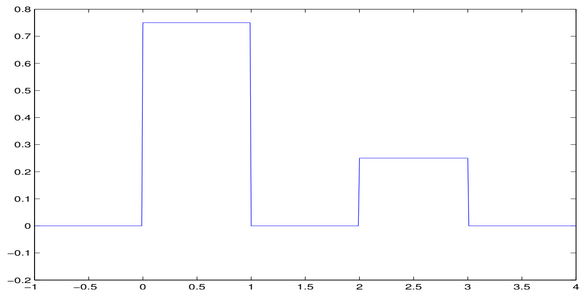

Figure 1: Probability density function f.

b. Determine the distribution function of F of X, and draw its graph. We first show how to compute the distribution function,

1. If $a < 0$, then

$$\begin{aligned} F(a) &= \int_{-\infty}^{a} 0 dx \\ &= 0 \end{aligned}$$

2. If $0 \leq a < 1$, then

$$\begin{aligned} F(a) &= \int_{-\infty}^{0} 0 dx + \int_{0}^{a} \frac{3}{4} dx \\ &= 0 + \left[\frac{3x}{4}\right]_0^a \\ &= \frac{3a}{4} \end{aligned}$$

3. If $1 \leq a < 2$, then

$$\begin{aligned} F(a) &= \int_{-\infty}^{0} 0dx + \int_{0}^{1} \frac{3}{4}dx + \int_{1}^{a} 0dx \\ &= 0 + \left[\frac{3x}{4}\right]_0^1 + 0 \\ &= \frac{3}{4} \end{aligned}$$

4. If $2 \leq a < 3$, then

$$\begin{aligned} F(a) &= \int_{-\infty}^{0} 0dx + \int_{0}^{1} \frac{3}{4}dx + \int_{1}^{2} 0dx + \int_{2}^{a} \frac{1}{4}dx \\ &= 0 + \left[\frac{3x}{4}\right]_0^1 + 0 + \left[\frac{x}{4}\right]_2^a \\ &= 0 + \frac{3}{4} + 0 + (\frac{a}{4} - \frac{2}{4}) \\ &= \frac{a+1}{4} \end{aligned}$$

5. If $3 \leq a$, then

$$\begin{aligned} F(a) &= \int_{-\infty}^{0} 0dx + \int_{0}^{1} \frac{3}{4}dx + \int_{1}^{2} 0dx + \int_{2}^{a} \frac{1}{4}dx + \int_{3}^{a} 0dx \\ &= 0 + \left[\frac{3x}{4}\right]_0^1 + 0 + \left[\frac{x}{4}\right]_2^3 + 0 \\ &= 0 + \frac{3}{4} + 0 + (\frac{3}{4} - \frac{2}{4}) + 0 \\ &= 1 \end{aligned}$$

We then plot the distribution function $F(X)$ in Figure 2.

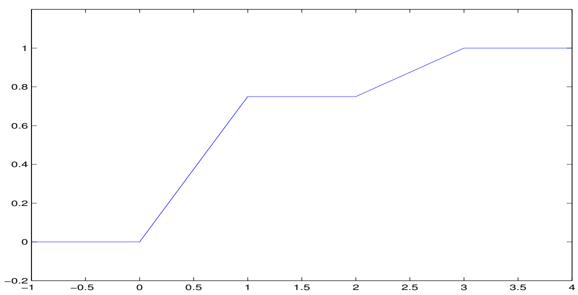

Figure 2: Distribution function $F(X)$.

5.5 The probability density function f of a continuous random variable X is given by:

$$f(x) = \begin{cases} cx + 3 & \text{for } -3 \leq x \leq -2 \\ 3 - cx & \text{for } 2 \leq x \leq 3 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Compute c.

There are two properties for a given probability density function f:

$$f(x) \geq 0, \text{ for } -\infty \leq x \leq \infty$$
$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\mathrm{d}x &= \int_{-\infty}^{-3} 0\mathrm{d}x + \int_{-3}^{-2} (cx+3)\mathrm{d}x + \int_{2}^{3} (3-cx)\mathrm{d}x + \int_{3}^{\infty} 0\mathrm{d}x \\ &= 0 + \left[\frac{cx^2}{2} + 3x\right]_{-3}^{-2} + 0 + \left[3x - \frac{cx^2}{2}\right]_{2}^{3} + 0 \\ &= 0 + \left(\left(\frac{c(-2)^2}{2} + 3(-2)\right) - \left(\frac{c(-3)^2}{2} + 3(-3)\right)\right) + 0 + \left(\left(3(3) - \frac{c3^2}{2}\right) - \left(3(2) - \frac{c2^2}{2}\right)\right) \\ &= 0 + \frac{4c}{2} - 6 - \frac{9c}{2} + 9 + 0 + 9 - \frac{9c}{2} - 6 + \frac{4c}{2} \\ &= 6 - 5c = 1 \\ &\implies \\ c &= 1 \end{aligned}$$

b. Compute the distribution function of X. Given $c = 1$, the probability density function is

$$f(x) = \begin{cases} x + 3 & \text{for } -3 \leq x \leq -2 \\ 3 - x & \text{for } 2 \leq x \leq 3 \\ 0 & \text{elsewhere.} \end{cases}$$

1. If $a < -3$, then

$$\begin{aligned} F(a) &= \int_{-\infty}^{a} 0\mathrm{d}x \\ &= 0 \end{aligned}$$

2. If $-3 \leq a < -2$, then

$$\begin{aligned}
F(a) &= \int_{-\infty}^{-3} 0 \mathrm{d}x + \int_{-3}^{a} (x+3) \mathrm{d}x \\
&= 0 + \left[\frac{x^2}{2} + 3x\right]_{-3}^{a} \\
&= 0 + \left(\frac{(a)^2}{2} + 3(a)\right) - \left(\frac{(-3)^2}{2} + 3(-3)\right) \\
&= 0 + (\frac{a^2}{2} + 3a) - (\frac{9}{2} - 9) \\
&= \frac{a^2 + 6a + 9}{2} \\
&= \frac{(a+3)^2}{2}
\end{aligned}$$

3. If $-2 \leq a < 2$, then

$$\begin{aligned}
F(a) &= \int_{-\infty}^{-3} 0 \mathrm{d}x + \int_{-3}^{-2} (x+3) \mathrm{d}x + \int_{-2}^{a} 0 \mathrm{d}x \\
&= 0 + \left[\frac{x^2}{2} + 3x\right]_{-3}^{-2} + 0 \\
&= 0 + \left(\frac{(-2)^2}{2} + 3(-2)\right) - \left(\frac{(-3)^2}{2} + 3(-3)\right) + 0 \\
&= 0 + (2-6) - (\frac{9}{2} - 9) + 0 \\
&= \frac{1}{2}
\end{aligned}$$

4. If $2 \leq a < 3$, then

$$\begin{aligned}
F(a) &= \int_{-\infty}^{-3} 0 \mathrm{d}x + \int_{-3}^{-2} (x+3) \mathrm{d}x + \int_{-2}^{2} 0 \mathrm{d}x + \int_{2}^{a} (3-x) \mathrm{d}x \\
&= 0 + \left[\frac{x^2}{2} + 3x\right]_{-3}^{-2} + 0 + \left[3x - \frac{x^2}{2}\right]_{2}^{a} \\
&= 0 + \left(\frac{(-2)^2}{2} + 3(-2)\right) - \left(\frac{(-3)^2}{2} + 3(-3)\right) + 0 + \left(\left(3a - \frac{a^2}{2}\right) - \left(3(2) - \frac{2^2}{2}\right)\right) \\
&= 0 + (2-6) - (\frac{9}{2} - 9) + 0 + \left(3a - \frac{a^2}{2} - 4\right) \\
&= 0 + \frac{1}{2} + 0 + \frac{6a - a^2 - 8}{2} \\
&= \frac{6a - a^2 - 7}{2}
\end{aligned}$$

5. If $3 < a$, then

$$\begin{aligned}
F(a) &= \int_{-\infty}^{-3} 0 \mathrm{d}x + \int_{-3}^{-2} (x+3) \mathrm{d}x + \int_{-2}^{2} 0 \mathrm{d}x + \int_{2}^{3} (3-x) \mathrm{d}x + \int_{3}^{a} 0 \mathrm{d}x \\
&= 0 + \left[\frac{x^2}{2} + 3x\right]_{-3}^{-2} + 0 + \left[3x - \frac{x^2}{2}\right]_{2}^{3} + 0 \\
&= 0 + \left(\frac{(-2)^2}{2} + 3(-2)\right) - \left(\frac{(-3)^2}{2} + 3(-3)\right) + 0 + \left(\left(3(3) - \frac{3^2}{2}\right) - \left(3(2) - \frac{2^2}{2}\right)\right) + 0 \\
&= 0 + (2-6) - (\frac{9}{2} - 9) + 0 + \left(3(3) - \frac{3^2}{2} - 4\right) + 0 \\
&= 0 + \frac{1}{2} + 0 + \frac{6 \times 3 - 3^2 - 8}{2} \\
&= 1
\end{aligned}$$

We then plot the distribution function $F(X)$ in Figure 3,

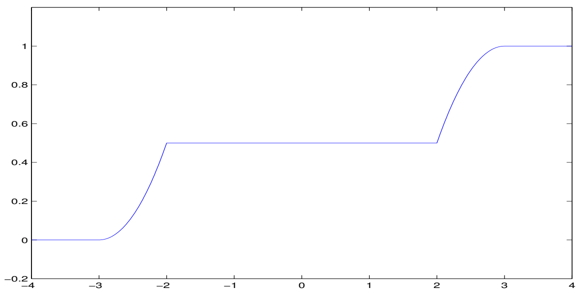

Figure 3: Distribution function $F(X)$.