Notes for Chapter 4

4.3-Bernoulli Distribution

Examples For a random variable X of a Bernoulli distribution, there are **TWO and ONLY TWO** values for X.

	Experiments	Outcomes	X=Success	X=Failure	PMF
Ber_1	Flip a coin	$\{$HEAD, TAIL$\}$	$\{$HEAD$\}$	$\{$TAIL$\}$	$Ber(\frac{1}{2})$
Ber_2	Flip a coin	$\{$HEAD, TAIL$\}$	$\{$TAIL$\}$	$\{$HEAD$\}$	$Ber(\frac{1}{2})$
Ber_3	Throw a dice	$\{$1, 2, 3, 4, 5, 6$\}$	$\{3\}$	$\{$1, 2, 4, 5, 6$\}$	$Ber(\frac{1}{6})$
Ber_4	Throw a dice	$\{$1, 2, 3, 4, 5, 6$\}$	Odd $\{$1, 3, 5$\}$	Even $\{$2, 4, 6$\}$	$Ber(\frac{1}{2})$
Ber_5	Throw a dice	$\{$1, 2, 3, 4, 5, 6$\}$	Even $\{$2, 4, 6$\}$	Odd $\{$1, 3, 5$\}$	$Ber(\frac{1}{2})$
Ber_6	Randomly pick a student	$\{$MALE, FEMALE$\}$	$\{$MALE$\}$	$\{$FEMALE$\}$	$Ber(\frac{19}{24})$
Ber_7	Randomly pick a student	$\{$MALE, FEMALE$\}$	$\{$FEMALE$\}$	$\{$MALE$\}$	$Ber(\frac{5}{24})$
Ber_1	Flip 2 coin	$\{$HH, HT,TH,TT$\}$	$\{$HH,TH,HT$\}$	$\{$TT$\}$	$Ber(\frac{3}{4})$

Table 1: Bernoulli distribution examples.

4.4-Binomial Distribution

Examples For a binomial distribution, multiple (n) Bernoulli trials with k success, each of which is with the probability p. If we try each of the experiments from Table 1 multiple times (n), we will get the following binomial distributions. (Note that those n repeats are independent of each other)

	X	One Bernoulli Trial (success)	n	p	PMF
Bin_1	Flip a coin 3 times with k heads, k=0,1,2,3	Flip a coin with a head	3	$\frac{1}{2}$	$Bin(3, \frac{1}{2})$
Bin_3	Flip 3 coins with k heads, k=0,1,2,3	Flip a coin with a head	3	$\frac{1}{2}$	$Bin(3, \frac{1}{2})$
Bin_4	Flip a coin 5 times with k tails,k=0,1,2,3,4,5	Flip a coin with a tail	5	$\frac{1}{2}$	$Bin(5, \frac{1}{2})$
Bin_5	Throw a dice 2 times with k threes,k=0,1,2	Throw a dice with a three	2	$\frac{1}{6}$	$Bin(2, \frac{1}{6})$
Bin_6	Throw a dice 3 times with k odd numbers,k=0,1,2,3	Throw a dice with an odd number	3	$\frac{1}{2}$	$Bin(3, \frac{1}{2})$
Bin_7	Throw a dice 4 times with k even numbers,k=0,1,2,3,4	Throw a dice with an even number	4	$\frac{1}{2}$	$Bin(4, \frac{1}{2})$
Bin_8	Randomly pick 2 students from the class with k male,k=0,1,2	Randomly pick a student from the class with a male	2	$\frac{19}{24}$	$Bin(2, \frac{19}{24})$
Bin_9	Randomly pick 5 students from the class with k female,k=0,1,2,3,4,5	Randomly pick a student from the class with a female	5	$\frac{5}{24}$	$Bin(5, \frac{5}{24})$

Table 2: Binomial distribution examples.

Exercise Given a dice with six numbers ($\{1, 2, 3, 4, 5, 6\}$), each number comes with the same probability when you roll it. Here is the game. Suppose you have such two dices and you simultaneously roll both of them to get the sum of the two output numbers. When the sum is 2 or 12, we say that you get the magic numbers and you will be rewarded. However, each play will cost you a certain amount of money and you can only afford to play n times. Let the random variable X denote the total number of times you will hit those magic numbers and be rewarded.

- What type of distribution does X have? Specify its parameter(s).
  Binomial distribution.
  A discrete random variable X has a *binomial distribution* with parameters n and p, where n=1, 2, ... and $0 \leq p \leq 1$,
  $$p_X(k) = P(X = k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k},$$
  for $k = 1, 2, \ldots, n$.
  There are **2** parameters, n and p^*, where n is the number of plays and p^* is the probability that you will hit the magic numbers and be rewarded in one single play such that $p^* = \frac{2}{36} = \frac{1}{18}$. (Two cases: 1+1 or 6 + 6)
- What is the probability mass function of the total number of plays X?
  Then we get
  $$p_X(k) = P(X = k) = \begin{pmatrix} n \\ k \end{pmatrix} (\frac{1}{18})^k (\frac{17}{18})^{n-k},$$

Geometric Distribution

Examples For a geometric distribution, succeed after k trials. In each trial, the probability of success is p. If we try each of the experiments from Table 1 repeatedly until succeed, we will get the following geometric distributions. (Note that those k repeats are independent of each other)

	X	One Success	p	PMF
Geo_1	Flip a coin k times until get a head,k=1,2,...	Flip a coin with a head	$\frac{1}{2}$	$Geo(\frac{1}{2})$
Geo_2	Flip a coin k times until get a tail,k=1,2,...	Flip a coin with a tail	$\frac{1}{2}$	$Geo(\frac{1}{2})$
Geo_3	Throw a dice k times until get a THREE,k=1,2,...	Throw a dice with a THREE	$\frac{1}{6}$	$Geo(\frac{1}{6})$
Geo_4	Throw a dice k times until get an odd number,k=1,2,...	Throw a dice with an odd number	$\frac{1}{2}$	$Geo(\frac{1}{2})$
Geo_5	Throw a dice k times until get an even number,k=1,2,...	Throw a dice with an even number	$\frac{1}{2}$	$Geo(\frac{1}{2})$
Geo_6	Randomly pick k student until get a male student,k=1,2,...	Randomly pick a student from the class with a male	$\frac{19}{24}$	$Geo(\frac{19}{24})$
Geo_7	Randomly pick k student until get a female student,k=1,2,...	Randomly pick a student from the class with a female	$\frac{5}{24}$	$Geo(\frac{5}{24})$

Table 3: Geometric distribution examples.

Exercise 1 (Same game here) Given a dice with six numbers ($\{1, 2, 3, 4, 5, 6\}$), each number comes with the same probability when you roll it. Suppose you have such two dices and you simultaneously roll both of them to get the sum of the two output numbers. When the sum is 2 or 12, we say that you get the magic numbers and you will be rewarded. You are so addicted to this game and will not stop until win it once (get 2 or 12 in one play). Let the random variable Y denote the number of plays when you stop playing.

- What type of distribution does Y have? Specify its parameter(s).
  Geometric distribution. It has one parameter p^* and it denotes the probability that the player hits those magic numbers and be rewarded such that $p^* = \frac{1}{18}$.

- What is the probability mass function of the random variable Y?
  Then we get

$$p_Y(k) = P(Y = k) = (\frac{17}{18})^{k-1}\frac{1}{18},$$

for $k = 1, 2, \ldots$.

Exercise 2 Let X have a $Geo(p)$ distribution. For $n \geq 0$, show that $P(X > n) = (1-p)^n$

$$\begin{aligned}
P(X > n) &= 1 - P(X \leq n) \\
&= 1 - \sum_{k=1}^{n} P(X = k) \\
&= 1 - \sum_{k=1}^{n} (1-p)^{k-1} p \\
&= 1 - p - (1-p)p - (1-p)^2 p - \ldots - (1-p)^{n-1} p \\
&= (1-p)^2 - (1-p)^2 p - \ldots - (1-p)^{n-1} p \\
&= (1-p)^n
\end{aligned}$$

In other words, $P(X > n)$ means that the the previous n times are all failures and the probability is $(1-p)^n$.

Exercise 3 For a geometric distribution $Geo(p)$, show that $P(X > n+k|X > k) = P(X > n)$ for $n, k = 0, 1, 2, \ldots$.

$$\begin{aligned}
P(X > n+k|X > k) &= \frac{P\left(\{X > n+k\} \cap \{X > k\}\right)}{P(X > k)} \\
&= \frac{P(X > n+k)}{P(X > k)} \\
&= \frac{(1-p)^{n+k}}{(1-p)^k} && (\,use\ Exercise\ 2) \\
&= (1-p)^n \\
&= P(X > n)
\end{aligned}$$

This is known as the *memoryless property*.

The property is most easily explained in terms of “waiting times.” Suppose that a random variable, X, is defined to be the time elapsed in a bank local branch from 9 am on a certain day until the arrival of the first customer: thus X is the time this local branch waits for the first customer. The “memoryless” property makes a comparison between the probability distributions of the time the local branch has to wait from 9 am onwards for his first customer, and the time that the local branch still has to wait for the first customer on those occasions when no customer has arrived by any given later time: the property of memorylessness is that these distributions of “**time from now to the next customer**” are exactly the same.
$P(X > n)$ means that the local branch has to wait for n time for the first customer.
$P(X > n+k|X > k)$ means that the local branch still has to wait for n time at any specific time point k when they still haven’t met the first customer.

Similarities and Dissimilaries

Distribution	Discrete	Trials	Success	Notation	PMF
Bernoulli	Yes	Single	0 or 1	$Ber(p)$	$p_X(1) = P(X = 1) = p$ $p_X(0) = P(X = 0) = 1 - p$
Binomial	Yes	Multiple	k	$Bin(n, p)$	$p_X(k) = P(X = k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k(1-p)^{(n-k)}$
Geometric	Yes	Multiple	1	$Geo(p)$	$p_X(k) = P(X = k) = (1-p)^{k-1}p$

Table 4: Summary.