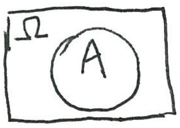
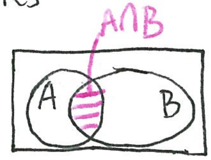
Chapter 3: Conditional Probability

1. Association of Area with Probabilities

① 

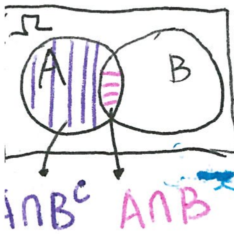
$P(A) = \dfrac{\text{Area}(A)}{\text{Area}(\Omega)}$

② 

$$P(A|B) = \frac{\text{Area}(A\cap B)}{\text{Area}(B)} = \frac{\text{Area}(A\cap B)/\text{Area}(\Omega)}{\text{Area}(B)/\text{Area}(\Omega)} = \frac{P(A\cap B)}{P(B)}$$

$\Rightarrow$ $P(A|B)$ describes given B happened, restrict attention to those elements of B that also belongs to A, which is $A\cap B$, what proportion of B is the collection of $A\cap B$?

2. $P(A) = P(A\cap B) + P(A\cap B^c)$



We can see, $\text{Area}(A) = \text{Area}(A\cap B) + \text{Area}(A\cap B^c)$

Divide $\text{Area}(\Omega)$ on both side:

$$\frac{\text{Area}(A)}{\text{Area}(\Omega)} = \frac{\text{Area}(A\cap B)}{\text{Area}(\Omega)} + \frac{\text{Area}(A\cap B^c)}{\text{Area}(\Omega)} \Rightarrow P(A) = P(A\cap B) + P(A\cap B^c)$$

Question 2:

$P(E^c|I)$?

$$\boxed{\begin{aligned} P(I^c|E^c) &= p \\ P(I|E) &= p \\ P(E) &= 7\% \end{aligned}}$$ Given

b) $\xrightarrow[\text{Probs}]{\text{flip the}}$

$$P(E^c|I) = \frac{P(E^c\cap I)}{P(I)}$$ ($P(E^c\cap I)$, $P(I)$ still unkown so keep fliping.)

$$= \frac{P(I|E^c)\cdot P(E^c)}{P(I)}$$ ($P(I|E^c)$, $P(E^c)$ given. why! $P(I)$ not given.)

$$= \frac{P(I|E^c)\cdot P(E^c)}{P(I\cap E^c) + P(I\cap E)}$$ ($P(I) = P(I\cap E) + P(I\cap E$

$$= \frac{P(I|E^c)\cdot P(E^c)}{P(I|E^c)\cdot P(E^c) + P(I|E)\cdot P(E)}$$ (Now, everything is given

$$= \frac{(1-p)\cdot 0.93}{(1-p)\cdot 0.93 + p\cdot 0.07} \approx \boxed{59.6\%}$$

c) $P(E|I) = 1 - P(E^c|I)$

$$= 1 - \frac{93(1-p)}{93-86p}$$

we want $P(E|I) = 0.9$

$\Rightarrow$ $0.9 = 1 - \dfrac{93-93p}{93-86p}$

$84.4p = 83.7$

$$\boxed{p \approx 99.2\%}$$

Q3: Event A: the suspect with the taature

Event B: the suspect is ~~the~~ guilt

Given: $P(A|B^c)=0.2 \quad P(A|B)=1$

$P(B)=0.6$

Question: $P(B|A)$?

Solution:

$$P(B|A)=\frac{P(A\cap B)}{P(A)} \quad \text{(conditional prob)}$$

$$=\frac{P(A|B)\cdot P(B)}{P(A)} \quad \text{(multiplication Rule)}$$

$$=\frac{1\times 0.6}{P(A|B^c)\cdot P(B^c)+P(A|B)\cdot P(B)} \quad \text{(total Prob)}$$

$$\approx 0.882$$

Q4. a). Given: $P(T|D)=0.95$, $P(T^c|D^c)=0.90$, $P(D)=0.01$

Quest: $P(D|T)$

$$P(D|T)=\frac{P(T|D)\cdot P(D)}{P(T|D)\cdot P(D)+P(T|D^c)\cdot P(D^c)} \quad \text{(Bayes Rule)}$$

$$=\frac{P(T|D)\cdot P(D)}{P(T|D)\cdot P(D)+(1-P(T^c|D^c))\cdot(1-P(D))} \quad \text{(Complement Rule)}$$

$$=\frac{0.01\times 0.95}{0.95\times 0.01+(1-0.9)\cdot(1-0.01)} \approx \boxed{0.0876}$$

b). Solution 1. One way is to replace the value of $P(D)$ with $P(D|T)$ from a).

$$P(D|T\cap S) \text{ or } P(D|T,S) = \frac{P(D|T)\cdot P(S|D)}{P(D|T)\cdot P(S|D)+(1-P(S^c|D^c))\cdot(1-P(D|T))} \quad \text{(Bayes + Complement}$$

S: second test said YES

$P(S|D)=P(T|D)$

$P(S^c|D^c)=P(T^c|D^c)$

$$=\frac{0.0876\times 0.95}{0.0876\times 0.95+(1-0.876)\cdot(1-0.9)} \approx \boxed{0.477}$$

Solution 2. Still compute $P(D|T\cap S)$

$$P(D|T\cap S)=\frac{P(D\cap T\cap S)}{P(T\cap S)} \quad \text{(conditional probs)}$$

$$=\frac{P(D)\times P(T\cap S|D)}{P(D)\cdot P(T\cap S|D)+P(D^c)\cdot P(T\cap S|D^c)} \quad \left(\frac{\text{Multiplication}}{\text{Total}}\right)$$

$$=\frac{P(D)\cdot \cancel{P(T\cap S|D)}\cdot P(T|D)\cdot P(S|D)}{P(D)\cdot P(T|D)\cdot P(S|D)+P(D^c)\cdot[1-P(T^c|D^c)]\cdot[1-P(S^c|D^c)]} \quad \text{(Independence}$$

$$=\frac{0.01\times 0.95\times 0.95}{0.01\times 0.95\times 0.95+(1-0.01)\cdot(1-0.90)\times(1-0.90)}$$

$$\approx \boxed{0.477}$$

Q5. Independence implies: $P(A|B)=P(A)$ and $P(B|A)=P(B)$ & $P(A\cap B)=P(A)$

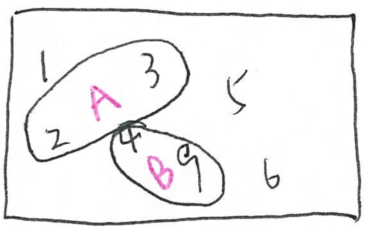
a). No. $\because P(A\cap B)=0$ $(A\cap B=\emptyset)$ $\Rightarrow A, B$ are not independent
$P(A)\cdot P(B)>0$

b). No. $\because A, B$ independent $\Rightarrow A\cap B\neq\emptyset \Rightarrow A, B$ are not disjoint
$\therefore P(A\cap B)=P(A)\cdot P(B)>0$

c). No. if $A\subset B$, $\Rightarrow P(B|A)=\dfrac{P(B\cap A)}{P(A)}=\dfrac{P(A)}{P(A)}=1$. But, $P(B)\in(0,1)\neq 1$
$\Rightarrow A, B$ are not independent.

d). No. $\because A\subset A\cup B$, $\Rightarrow P(A\cup B|A)=\dfrac{P((A\cup B)\cap A)}{P(A)}=\dfrac{P(A)}{P(A)}=1$
while $P(A\cup B)=P(A)+P(B)-P(A\cap B)$, (union Rule)
$=P(A)+P(B)-P(A)\cdot P(B)$, ($\because A, B$ independent)
$\Rightarrow$

Eg. For (a). if $\Omega=\{1, 2, 3, 4, 5, 6, 9\}$. Event A contains $\{2, 3\}$. Every element in B event is the square of the elements in A, so $B=\{4, 9\}$.



A, B are disjoint $\Rightarrow A\cap B=\emptyset$.

But A, B are dependent. because

$P(B=4|A=2)=1$, $P(B=6|A=2)=0$

while $P(B=4)=\dfrac{1}{7}$ $P(B=6)=\dfrac{1}{7}$.

Once we know A, B is fixed.