Chapter 3
Conditional Probability and Independence

1 Summary

- The *conditional probability* of A given C is

$$P(A|C) = \frac{P(A \cap C)}{P(C)}, P(C) > 0 \tag{1}$$

- The *multiplication rule*. For any events A and C,

$$\begin{aligned} P(A \cap C) &= P(C) \cdot P(A|C) &(2)\\ &= P(A) \cdot P(C|A) &(3) \end{aligned}$$

- The *law of total probability*. Suppose $C_1, C_2, \ldots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup \cdots \cup C_m = \Omega$. The probability of an arbitrary event A can be expressed as:

$$P(A) = P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + \cdots + P(A|C_m)P(C_m) \tag{4}$$

- *Bayes' rule*. Suppose the events $C_1, C_2, \ldots, C_m$ are disjoint and $C_1 \cup C_2 \cup \cdots \cup C_m = \Omega$. The conditional probability of C_i, given an arbitrary event A, can be expressed as:

$$P(C_i|A) = \frac{P(A|C_i) \cdot P(C_i)}{P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + \cdots + P(A|C_m)P(C_m)} \tag{5}$$

- Independence vs Dependence
  - An event A is called independent of B if

$$P(A|B) = P(A) \tag{6}$$

  - To show that A and B are independent it suffices to prove just one of the following:

$$\begin{aligned} P(A|B) =& P(A) &(7)\\ P(B|A) =& P(B) &(8)\\ P(A \cap B) =& P(A)P(B) &(9) \end{aligned}$$

    where A may be replaced by A^c and B replaced by B^c, or both. If one of these statements holds, all of them are true. If two events are not independent, they are dependent.
  - Independence of two or more events. Events $A_1, A_2, \ldots, A_m$ are called independent if

$$P(A_1 \cap A_2 \cap A_3 \cdots A_m) = P(A_1)P(A_2) \cdots P(A_m) \tag{10}$$

    and this statement also holds when any number of the events $A_1, A_2, \ldots, A_m$ are replaced by their complements throught the formula.