Homework based on Chapter 20, 21
Computational Probability and Statistics
CIS 2033, Section 002

Due: 9:00 AM, Friday, April 24, 2015

20.1 (5 points) Given a random sample $X_1, X_2, \ldots, X_n$ from a distribution with finite variance σ^2. We estimate the expectation of the distribution with the sample mean $\overline{X}_n$. Argue that the larger our sample, the more efficient our estimator. What is the relative efficiency $Var(\overline{X}_n)/Var(\overline{X}_{2n})$ of $\overline{X}_{2n}$ with respect to $\overline{X}_n$?

Answer: Since, X_i and X_j are independent, then $Cov(X_i, X_j) = 0$ for $i \neq j$. Let $Var(X_i) = \sigma^2, i = 1, 2, \ldots, n$, $Cov(X_i, X_j) = \gamma = 0, i \neq j$. Then $Var(\overline{X}_n) = Var(\frac{X_1+X_2+\ldots+X_n}{n}) = \frac{1}{n^2}\left(n\sigma^2 + n(n-1)\gamma\right) = \frac{\sigma^2}{n}$. Then, the larger n is, the smaller $Var(\overline{X}_n)$ is, the more efficient the estimator is. (Also means that the smaller MSE is).

$\frac{Var(\overline{X}_n)}{Var(\overline{X}_{2n})} = \frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{2n}} = 2$, which means that $\overline{X}_{2n}$ is twice as efficient as $\overline{X}_n$.

20.7 (5 points) In Exercise 19.7 you showed that both T_1 and T_2 are unbiased estimators for θ. Which estimator would you prefer ? Motivate your answer.

Answer: $Var(T_1) = \frac{16}{n^2}Var(N_1) = \frac{16}{n^2}np_1(1-p_1) = \frac{16}{n}\frac{1}{4}(\theta+2)\frac{1}{4}(2-\theta) = \frac{4-\theta^2}{n}$
$Var(T_2) = \frac{16}{n^2}Var(N_2) = \frac{16}{n^2}np_2(1-p_2) = \frac{16}{n}\frac{\theta}{4}(1-\frac{\theta}{4}) = \frac{4\theta-\theta^2}{n}$
Since $0 < \theta < 1$, then $\frac{4\theta-\theta^2}{n} < \frac{4-\theta^2}{n}$. T_2 is more efficient.

21.9 (5 points) Tossing a coin is a Bernoulli trial, the random variable X denoting the results of a toss follows a Bernoulli distribution $\sim Ber(p)$, where p is the probability of getting a head.

a). Suppose you observed 10 tosses of a coin which are H,T,T,T,T,H,H,T,T,T. Determine the maximum likelihood estimates for p.

Answer: For $X \sim Ber(p)$ if $P(X = H) = p, P(X = T) = 1 - p$ Then, the likelihood for the observation is a function in terms of p, $L(p) = p^3 * (1-p)^7$. To achieve the maximum of $L(p)$, we have to set $\frac{dL(p)}{dp} = \frac{d(p^3)}{dp} * (1-p)^7 + p^3 * \frac{d(1-p)^7}{dp} = 3 * p^2 - 7p^3(1-p)^6 = 0$, which we get the maximum likelihood estimator for $p = \frac{3}{10} = 0.3$.

21.10 (5 points) Let $x_1, x_2, \ldots, x_n$ be a dataset that are observations of a random variable from a $Par(\alpha)$ distribution. What is the maximum likelihood estimate for α.

Answer: For $Par(\alpha)$, $p(x_i) = \frac{\alpha}{x_i^{\alpha+1}}$, then $L(\alpha) = \prod_{i=1}^n \frac{\alpha}{x_i^{\alpha+1}} = \frac{\alpha^n}{\prod x_i^{\alpha+1}}$, then $\ell(\alpha) = n\log(\alpha) - (\alpha+1)\sum_{i=1}^n \log(x_i)$, let the derivative to be zero, then $\frac{n}{\alpha} - \sum_{i=1}^n \log x_i = 0$, then $\alpha = \frac{n}{\sum_{i=1}^n \log x_i}$