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Homework 7

1. a) X

a	-1	0	1
$p(a)$	$1/4$	$2/4$	$1/4$
$a \cdot p(a)$	$-1/4$	0	$1/4$

$\Sigma = 0 = E[X]$

$Y = X^2$

b	0	1
$p(b)$	$2/4$	$2/4$
$b \cdot p(b)$	0	$2/4$

$\Sigma = 1/2 = E[Y]$

		a			
		-1	0	1	$P(Y=b)$
b	0	0	$2/4$	0	$2/4$
	1	$1/4$	0	$1/4$	$2/4$
$P(X=a)$		$1/4$	$2/4$	$1/4$	1

b) $Cov(X,Y) = E[XY] - E[X]E[Y]$

$E[XY] = (-1)(1)(1/4) + (0)(0)(2/4) + (1)(1)(1/4)$
$= -1/4 + 0 + 1/4$
$= 0$

$Cov(X,Y) = 0 - (0)(1/2)$
$\boxed{Cov(X,Y) = 0}$

c) $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \dfrac{0}{\sqrt{Var(X)\,Var(Y)}}$

$\boxed{\rho(X,Y) = 0}$
$\boxed{X \text{ and } Y \text{ are uncorrelated}}$

d) $\boxed{X \text{ and } Y \text{ are dependent}}$
$P(X=-1, Y=0) = 0 \neq P(X=-1)P(Y=0) = (1/4)(2/4) = 1/8$

2. a)

b \ a	-1	0	1
4	$\lambda - \frac{1}{16}$	λ	0
5	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{4}$
6	λ	$\frac{1}{8}$	$\frac{1}{4} - \lambda$

$\lambda - \frac{1}{16} + \lambda + 0 + \frac{1}{8} + \frac{1}{16} + \frac{1}{4} + \lambda + \frac{1}{8} + \frac{1}{4} - \lambda = 1$

$2\lambda + \frac{2}{16} + \frac{4}{16} + \frac{2}{16} + \frac{4}{16} = 1$

$2\lambda + \frac{12}{16} = \frac{16}{16}$

$2\lambda = \frac{4}{16} = \frac{1}{4}$ $\quad \boxed{\lambda = \frac{1}{8}}$

b)

b \ a	-1	0	1	P(Y=b)
4	$\frac{1}{16}$	$\frac{2}{16}$	0	$\frac{3}{16}$
5	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{7}{16}$
6	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{6}{16}$
P(X=a)	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{6}{16}$	1

c) $E[X] = (-1)\left(\frac{5}{16}\right) + (0)\left(\frac{5}{16}\right) + (1)\left(\frac{6}{16}\right) = -\frac{5}{16} + \frac{6}{16} = \boxed{\frac{1}{16}}$

$E[Y] = (4)\left(\frac{3}{16}\right) + (5)\left(\frac{7}{16}\right) + (6)\left(\frac{6}{16}\right) = \frac{12}{16} + \frac{35}{16} + \frac{36}{16} = \boxed{\frac{83}{16}}$

$Var(X) = E[X^2] - (E[X])^2 = \frac{11}{16} - \left(\frac{1}{16}\right)^2 = \frac{176}{256} - \frac{1}{256} = \boxed{\frac{175}{256}}$

$E[X^2] = (-1)^2\left(\frac{5}{16}\right) + (0)^2\left(\frac{5}{16}\right) + (1)^2\left(\frac{6}{16}\right) = \frac{5}{16} + \frac{6}{16} = \frac{11}{16}$

$Var(Y) = E[Y^2] - (E[Y])^2 = \frac{439}{16} - \left(\frac{83}{16}\right)^2 = \frac{7024}{256} - \frac{6889}{256} = \boxed{\frac{135}{256}}$

$E[Y^2] = (4)^2\left(\frac{3}{16}\right) + (5)^2\left(\frac{7}{16}\right) + (6)^2\left(\frac{6}{16}\right) = \frac{48}{16} + \frac{175}{16} + \frac{216}{16} = \frac{439}{16}$

d) $E[XY] = (-1)(4)\left(\frac{1}{16}\right) + (0)(4)\left(\frac{2}{16}\right) + (1)(4)(0) + (-1)(5)\left(\frac{2}{16}\right) + (0)(5)\left(\frac{1}{16}\right)$
$+ (1)(5)\left(\frac{4}{16}\right) + (-1)(6)\left(\frac{2}{16}\right) + (0)(6)\left(\frac{2}{16}\right) + (1)(6)\left(\frac{2}{16}\right)$

$= -\frac{4}{16} - \frac{10}{16} + \frac{20}{16} - \frac{12}{16} + \frac{12}{16}$

$= \frac{6}{16} = \boxed{\frac{3}{8}}$

$E[X]E[Y] = \left(\frac{1}{16}\right)\left(\frac{83}{16}\right) = \boxed{\frac{83}{256}}$

$\boxed{E[XY] \neq E[X]E[Y]}$

e) $E[X+Y] = (-1+4)(\frac{1}{16}) + (0+4)(\frac{2}{16}) + (1+4)(0) + (-1+5)(\frac{2}{16}) + (0+5)(\frac{1}{16}) +$
$(1+5)(\frac{4}{16}) + (-1+6)(\frac{2}{16}) + (0+6)(\frac{2}{16}) + (1+6)(\frac{2}{16})$

$= \frac{3}{16} + \frac{8}{16} + \frac{8}{16} + \frac{5}{16} + \frac{24}{16} + \frac{10}{16} + \frac{12}{16} + \frac{14}{16}$

$= \frac{84}{16} = \boxed{\frac{21}{4}}$

$E[X] + E[Y] = \frac{1}{16} + \frac{83}{16} = \frac{84}{16} = \boxed{\frac{21}{4}}$

$\boxed{E[X+Y] = E[X] + E[Y]}$

f) $Cov(X,Y) = E[XY] - E[X]E[Y]$

$= \frac{3}{8} - \frac{83}{256}$

$= \frac{96}{256} - \frac{83}{256}$

$= \boxed{\frac{13}{256}}$

$\rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$

$= \dfrac{(\frac{13}{256})}{\sqrt{\frac{175}{256} \cdot \frac{135}{256}}}$

$= \boxed{0.0846}$

X and Y are positively correlated

g) $P(X=1, Y=4) = 0 \qquad P(X=1) = \frac{6}{16} \qquad P(Y=4) = \frac{3}{16}$

$0 \neq (\frac{6}{16})(\frac{3}{16})$

X and Y are dependent

extra credit:

$Var(XY) = E[(XY)^2] - (E[XY])^2 = \frac{310}{16} - (\frac{3}{8})^2 = \frac{1240}{64} - \frac{9}{64} = \boxed{\frac{1231}{64} = 19.234}$

$E[XY] = \frac{3}{8}$

$E[(XY)^2] = (-1 \cdot 4)^2(\frac{1}{16}) + (0 \cdot 4)^2(\frac{2}{16}) + (1 \cdot 4)^2(0) + (-1 \cdot 5)^2(\frac{2}{16})$
$+ (0 \cdot 5)^2(\frac{1}{16}) + (1 \cdot 5)^2(\frac{4}{16}) + (-1 \cdot 6)^2(\frac{2}{16}) + (0 \cdot 6)^2(\frac{2}{16})$
$+ (1 \cdot 6)^2(\frac{2}{16})$

$= 16(\frac{1}{16}) + 0 + 0 + 25(\frac{2}{16}) + 0 + 25(\frac{4}{16}) + 36(\frac{2}{16}) + 0 + 36(\frac{2}{16})$

$= \frac{16}{16} + \frac{50}{16} + \frac{100}{16} + \frac{72}{16} + \frac{72}{16}$

$= \frac{310}{16} = 19.375$

$$\mathrm{Var}(X) \cdot \mathrm{Var}(Y) = \left(\frac{175}{256}\right)\left(\frac{135}{256}\right) = \frac{23625}{65536} = 0.360$$

$$\boxed{\mathrm{Var}(XY) \neq \mathrm{Var}(X)\,\mathrm{Var}(Y)}$$

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y) \\ &= \left(\frac{175}{256}\right) + \left(\frac{135}{256}\right) + 2\left(\frac{13}{256}\right) = \frac{336}{256} = \boxed{\frac{21}{16}}\end{aligned}$$

since $\mathrm{Cov}(X,Y) \neq 0$, $\boxed{\mathrm{Var}(X+Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y)}$